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TWO-DIMENSIONAL DYNAMIC BOUNDARY-VALUE PROBLEMS FOR CURVILINEAR THERMOVISCOELASTIC BODIES

A. D. Chernyshov

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A new numerical-analytical method is proposed and demonstrated using an example of dynamic problems of a thermoviscoelastic body. In the general formulation, the thermoviscoelastic problem is split into three simpler problems. In the first problem, boundary functions that should satisfy only boundary conditions are determined. The second problem with homogeneous boundary conditions and inhomogeneous initial conditions is reduced to an eigenvalue problem by introducing special ξ variables and separating time. This problem is solved by organizing integral superpositions with respect to the angular parameter. A linear algebraic system is obtained as a result of satisfaction of the boundary conditions at points that partition the curvilinear boundary of the body into small segments. After the eigenfunctions and eigenvalues are determined, the third problem with homogeneous boundary and initial conditions is solved by spectral decomposition of unknown functions and inhomogeneous terms in a coupled system of ordinary differential equations.

Key words: *thermoviscoelastic medium, dynamic boundary-value problem, eigenfunctions, eigenvalues.*

Introduction. Various models of a thermoelastic body were proposed in [1–3] and other papers. For geometrically multidimensional cases, the model of a thermoviscoelastic body is very complex since, in addition to elastic and temperature properties, it takes into account viscous characteristics. Therefore, analytical studies for these materials are few in number.

In the present paper, we restrict our consideration to a linear model. To this end, the quadratic terms of the strain-rate tensor which characterize energy dissipation due to viscosity are ignored in the heat-conduction equation. This simplification is allowable for times from the onset of loading such that the thermal accumulation of the indicated dissipation is small.

Existing methods for solving thermoelastic problems are based primarily on potentials [4] and integral transformations [5, 6]. The Ritz method [7], Bubnov–Galerkin method [1], variational principles [8], and finite-difference techniques [8, 9] are also used. The effectiveness of these methods is limited in the cases where the body has a complex curvilinear shape. Therefore, there have been very few studies in which the initial boundary-value problems are solved even for simple geometries of thermoelastic bodies. The present paper proposes a numerical-analytical method for solving these problems.

1. Formulation of the Problem. By a thermoviscoelastic body is meant a thermoelastic body with viscous properties. These properties are typical of metals and their alloys subjected to small mechanical and thermal loads. Figure 1 shows a rheological model of this body. In this case, the total stress σ_{ij} is decomposed into elastic $\sigma_{ij}^{(e)}$ and viscous $\sigma_{ij}^{(v)}$ components:

$$\sigma_{ij} = \sigma_{ij}^{(e)} + \sigma_{ij}^{(v)}, \qquad \sigma_{ij}^{(v)} = \zeta \varepsilon_{kk}\, \delta_{ij} + 2\chi\, \varepsilon_{ij}. \tag{1}$$

Voronezh State Technological Academy, Voronezh 394000; kafvm@vgta.vrn.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 46, No. 2, pp. 158–169, March–April, 2005. Original article submitted September 3, 2003; revision submitted March 24, 2004.

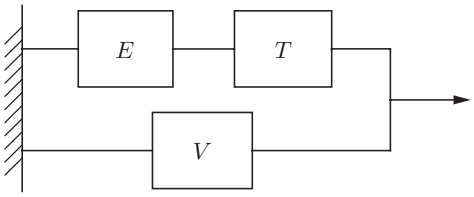

Fig. 1

Here ζ and χ are the viscosity coefficients. Using the laws of thermodynamics of irreversible processes and proceeding as in [4], we obtain the following linear equations of motion in displacements in Cartesian coordinates:

$$(\lambda+2\mu)u_{xx}+(\lambda+\mu)v_{xy}+\mu u_{yy}+(\zeta+2\chi)u_{txx}+(\zeta+\chi)v_{txy}+\chi u_{tyy}-\gamma T_x+\tilde{G}_1=\rho u_{tt}.$$

$$(\lambda+2\mu)v_{yy}+(\lambda+\mu)u_{xy}+\mu v_{xx}+(\zeta+2\chi)v_{tyy}+(\zeta+\chi)u_{txy}+\chi v_{txx}-\gamma T_y+\tilde{G}_2=\rho v_{tt}, \tag{2}$$

$$\gamma=(3\lambda+2\mu)\alpha_t,$$

Here λ and μ are the elastic constants, ρ is the density, $\tilde{G}_1$ and $\tilde{G}_2$ are the mass forces, and α_t is the thermal expansion coefficient. The differential equations of system (2) differ qualitatively from similar equations for a thermoelastic body in that each equation contains a third-order derivative at the viscosity coefficients. This complicates the solution of the equations. The heat-conduction equation now contains a quadratic term with the strain-rate tensor ε_{ij} due to viscous dissipation. For small ε_{ij}, this term can be omitted and this equation takes the same form as that for a thermoelastic body:

$$a\Delta T-\eta(u_{xt}+v_{yt})+\tilde{q}=T_t, \qquad \eta=\gamma T_0/(C\rho). \tag{3}$$

Here Δ is the Laplace operator, T_0 is the initial absolute temperature of the body, C is the specific heat, and $\tilde{q}$ is the internal heat source. It should be noted that although the omitted viscous dissipation is small, it can be accumulated with time and have considerable effect on the stress–strain state. Therefore, the closed linear system (2), (3) for u, v, and T is valid only for a certain finite interval of time t_0.

Let the body shape be a bounded domain Ω with a piecewise smooth boundary Γ. The boundary and initial conditions are written as

$$u\Big|_\Gamma=F_1(t,x_\Gamma,y_\Gamma), \qquad v\Big|_\Gamma=F_2(t,x_\Gamma,y_\Gamma), \qquad T\Big|_\Gamma=F_0(t,x_\Gamma,y_\Gamma); \tag{4}$$

$$u\Big|_{t=0}=\tilde{\varphi}_1(x,y), \qquad u_t\Big|_{t=0}=\tilde{\psi}_1(x,y),$$

$$v\Big|_{t=0}=\tilde{\varphi}_2(x,y), \qquad v_t\Big|_{t=0}=\tilde{\psi}_2(x,y), \qquad T\Big|_{t=0}=\tilde{\varphi}_0(x,y). \tag{5}$$

To solve problem (2)–(5), we write the unknowns u, v, and T as the sum of three functions of x, y, and t:

$$u=M_1+u^{(1)}+v^{(1)}, \qquad v=M_2+u^{(2)}+v^{(2)}, \qquad T=M_0+u^{(0)}+v^{(0)}. \tag{6}$$

The functions M_b $(b=0,1,2)$ will be called boundary functions and constructed so that boundary conditions (4) be satisfied. The functions M_b may not satisfy Eqs. (2) and (3) but should be three times differentiable with respect to all its variables, i.e.,

$$M_b\Big|_\Gamma=F_b(t,x_\Gamma,y_\Gamma), \qquad M_b\in C^3(\Omega,\ 0\leqslant t\leqslant t_0), \qquad b=0,1,2. \tag{7}$$

Let $u^{(0)}$, $u^{(1)}$, and $u^{(2)}$ satisfy the uncoupled homogeneous system

$$a\Delta u^{(0)}=u_t^{(0)}, \qquad (\lambda+2\mu)u_{xx}^{(1)}+\mu u_{yy}^{(1)}=\rho u_{tt}^{(1)}, \qquad (\lambda+2\mu)u_{yy}^{(2)}+\mu u_{xx}^{(2)}=\rho u_{tt}^{(2)} \tag{8}$$

subject to the homogeneous boundary conditions

$$u^{(b)}\Big|_\Gamma=0, \qquad b=0,1,2 \tag{9}$$

and the inhomogeneous initial conditions

$$u^{(b)}\Big|_{t=0} = \tilde{\varphi}_b - M_b\Big|_{t=0} = \varphi_b, \quad b = 0,1,2, \qquad u_t^{(c)}\Big|_{t=0} = \tilde{\psi}_c - M_{ct}\Big|_{t=0} = \psi_c, \quad c = 1,2. \tag{10}$$

Solving problem (8)–(10), we obtain the eigenfunctions and eigenvalues, which are then used to find $v^{(b)}$, i.e., the solution of the entire problem. Substituting u, v, and T from (6) into (2) and (3) and taking into account (8) for $v^{(b)}$ $(b = 0,1,2)$, we arrive at the following inhomogeneous coupled system of differential equations:

$$\begin{gathered}
a\Delta v^{(0)} - \eta(v_{tx}^{(1)} + v_{ty}^{(2)}) + q = v_t^{(0)},\\
q = \tilde{q} + a\Delta M_0 - \eta(M_{1tx} + M_{2ty} + u_{tx}^{(1)} + u_{ty}^{(2)}) - M_{0t},\\
(\lambda+2\mu)v_{xx}^{(1)} + (\lambda+\mu)v_{xy}^{(2)} + \mu v_{yy}^{(1)} - \gamma v_x^{(0)} + (\zeta+2\chi)v_{txx}^{(1)} + (\zeta+\chi)v_{txy}^{(2)} + \chi v_{tyy}^{(1)} + G_1 = \rho v_{tt}^{(1)},\\
(\lambda+2\mu)v_{yy}^{(2)} + (\lambda+\mu)v_{xy}^{(1)} + \mu v_{xx}^{(2)} - \gamma v_y^{(0)} + (\zeta+2\chi)v_{tyy}^{(2)} + (\zeta+\chi)v_{txy}^{(1)} + \chi v_{txx}^{(2)} + G_2 = \rho v_{tt}^{(2)},\\
G_1 = \tilde{G}_1 + (\lambda+2\mu)M_{1xx} + (\lambda+\mu)(M_{2xy} + u_{xy}^{(2)}) + \mu M_{1yy} + (\zeta+2\chi)(M_{1txx} + u_{txx}^{(1)})\\
+ (\zeta+\chi)(M_{2txy} + u_{txy}^{(2)}) + \chi(M_{1tyy} + u_{tyy}^{(1)}) - \gamma(M_{0x} + u_x^{(0)}) - \rho M_{1tt},\\
G_2 = \tilde{G}_2 + (\lambda+2\mu)M_{2yy} + (\lambda+\mu)(M_{1xy} + u_{xy}^{(1)}) + \mu M_{2xx} + (\zeta+2\chi)(M_{2tyy} + u_{tyy}^{(2)})\\
+ (\zeta+\chi)(M_{1txy} + u_{txy}^{(1)}) + \chi(M_{2txx} + u_{txx}^{(2)}) - \gamma(M_{0y} + u_y^{(0)}) - \rho M_{2tt}.
\end{gathered} \tag{11}$$

Using (4) and taking into account (6), (7), and (9), for system (11) for the unknowns $v^{(b)}$ $(b = 0,1,2)$, we obtain the homogeneous boundary conditions

$$v^{(b)}\Big|_\Gamma = 0, \qquad b = 0,1,2, \tag{12}$$

and from (5) and (10), we have the homogeneous initial conditions

$$v^{(b)}\Big|_{t=0} = 0, \quad b = 0,1,2; \qquad v_t^{(c)}\Big|_{t=0} = 0, \quad c = 1,2. \tag{13}$$

2. Finding the Boundary Functions M_b. The functions M_b should be found explicitly to use the method described below. The problem of the continuation of the functions F_b with Γ into the internal points of Ω was thoroughly studied by Nikol'skii [10]. We give some examples of the continuation.

The simplest case and the most important case is the one where the boundary conditions do not depend on the points of the boundary Γ and depend only on t. Then, $M_b = F_b(t)$. Let the functions F_b depend on the points of the boundary Γ. If boundary conditions (7) are supplemented by, for example, the Laplace equation $\Delta M_b = 0$, we express M_b in integral form in terms of Green's function [11]

$$M_b = \int_\Gamma F_b \frac{\partial G}{\partial n}\, dl, \qquad M_b\Big|_\Gamma = F_b. \tag{14}$$

The functions M_b can also be obtained using R-functions and Boolean algebra [12].

3. Organizing Integral Superpositions and Constructing Eigenfunctions. We first consider the auxiliary problem (8)–(10) for a specified domain Ω and $0 \leqslant t \leqslant t_0$. We seek particular solutions of system (8) ignoring the initial conditions. To this end, we separate the time variable in the functions $u^{(b)}$ $(b = 0,1,2)$ as follows:

$$u^{(c)} = (A_c \cos\nu_c t + B_c \sin\nu_c t)U_c(x,y), \quad c = 1,2; \qquad u^{(0)} = A_0\, \mathrm{e}^{-\nu_0^2 at}\, U_0(x,y). \tag{15}$$

Here A_0, A_c, and B_c $(c = 1,2)$ are unknown constants, ν_b $(b = 0,1,2)$ are eigenvalues, and U_b are eigenfunctions. To find them, we substitute $u^{(b)}$ from (15) into (8) and (9). Cancelling the time functions relative to U_b, we obtain the following system of uncoupled and independent homogeneous equations subject to homogeneous boundary conditions:

$$\Delta U_0 + \nu_0^2 U_0 = 0, \qquad U_b|_\Gamma = 0, \qquad b = 0,1,2,$$

$$(\lambda+2\mu)U_{1xx}+\mu U_{1yy}+\rho\nu_1^2U_1=0,\qquad (\lambda+2\mu)U_{2yy}+\mu U_{2xx}+\rho\nu_2^2U_2=0. \tag{16}$$

Finding ν_b $(b=0,1,2)$ such that nontrivial solutions U_b of system (16) exist for a specified domain Ω is called the eigenfunctions and eigenvalues problem. These boundary-value problems are usually solved using the Ritz and Bubnov–Galerkin methods, integral transformations, and finite-difference methods. We propose the following numerical-analytical method.

In the domain Ω, we choose a point O with radius vector $\boldsymbol{r}_0$ as a pole, so that any radial E drawn through O at any angle θ to the x axis intersects the boundary Γ only at two points D^+ and D^-. For each of the three equations of system (16), we introduce three new geometrical variables $\xi^{(b)}$ $(b=0,1,2)$ as follows:

$$\begin{gathered}\xi^{(b)}=a_b(x-x_0)\cos\theta+c_b(y-y_0)\sin\theta,\qquad (x,y)\in\Omega,\quad b=0,1,2,\\ a_0=c_0=1,\qquad a_1=c_2=\sqrt{\rho/(\lambda+2\mu)},\qquad a_2=c_1=\sqrt{\rho/\mu}\,.\end{gathered} \tag{17}$$

In (17), $\boldsymbol{n}=(\cos\theta,\sin\theta)$ is the unit vector directed along E.

Ignoring the boundary conditions, we find particular solutions of system (16) for the case where U_b depend only on one variable $\xi^{(b)}$. Setting $U_b=P(\xi^{(b)})$, we write all three equations from (16) as

$$P''(\xi^{(b)})+\nu_b^2P(\xi^{(b)})=0,\qquad b=0,1,2. \tag{18}$$

The general solution of Eqs. (18) is given by

$$P(\xi^{(b)})=K_b\cos(\nu_b\xi^{(b)})+L_b\sin(\nu_b\xi^{(b)}),\qquad b=0,1,2, \tag{19}$$

where K_b and L_b $(b=0,1,2)$ are arbitrary constants, that depend on the angular parameter θ. Upon variation of θ within the range $[0,\pi)$, the sets of points of intersection D^+ and D^- form the boundaries Γ^+ and Γ^-, respectively. Thus, the boundary Γ consists of two parts Γ^+ and Γ^-. For various values of θ, we obtain a bundle of radials E at the point O. For $\theta=\pi$, the location of the straight line E coincides with its location for $\theta=0$. If the straight line E is rotated within the limits $\theta\in[\pi,2\pi)$, the points D^+ and D^- pass through the boundary Γ for the second time, which is not needed. These considerations are used to choose the limits of variation in θ in the definite integrals and sums given below. The functions $P(\xi^{(b)})$ $(b=0,1,2)$ from (19) satisfy system (16) identically for any θ; therefore, using solutions (19), we construct the following solution of system (16) by integral superposition for θ:

$$\begin{gathered}U_b=\int\limits_0^\pi[K_b(\theta)\cos(\nu_b\xi^{(b)})+L_b(\theta)\sin(\nu_b\xi^{(b)})]\,d\theta+\sum_{i=1}^n[K_{bi}^*\cos(\nu_b\xi_i^{(b)})+L_{bi}^*\sin(\nu_b\xi_i^{(b)})],\\ \xi_i^{(b)}=a_b(x-x_0)\cos\theta_i+c_b(y-y_0)\sin\theta_i,\qquad 0\leqslant\theta_i<\pi,\qquad b=0,1,2,\qquad i=1,\ldots,n.\end{gathered} \tag{20}$$

Here $K_b(\theta)$ and $L_b(\theta)$ are unknown functions of θ summable in the Lebesgue sense, K_{bi}^* and L_{bi}^* and θ_i are unknown coefficients and angles in the finite sums, respectively, and n is an unknown number of terms. The presence of the finite sum of n terms in the eigenfunctions (20) makes the future expression of the solution more general because in some problems [13] the solution consists only of such terms; in other cases (for example, for a circle), it consists only of definite integrals over the parameter θ. To find the above-listed unknowns, we substitute U_b from (20) into the boundary conditions from (16):

$$\begin{gathered}U_b\Big|_\Gamma=\int\limits_0^\pi[K_b(\theta)\cos(\nu_b\xi_\Gamma^{(b)})+L_b(\theta)\sin(\nu_b\xi_\Gamma^{(b)})]\,d\theta+\sum_{i=1}^n[K_{bi}^*\cos(\nu_b\xi_{i\Gamma}^{(b)})+L_{bi}^*\sin(\nu_b\xi_{i\Gamma}^{(b)})]=0,\\ \xi_\Gamma^{(b)}=a_b(x_\Gamma-x_0)\cos\theta+c_b(y_\Gamma-y_0)\sin\theta,\qquad (x_\Gamma,y_\Gamma)\in\Gamma,\\ \xi_{i\Gamma}^{(b)}=a_b(x_\Gamma-x_0)\cos\theta_i+c_b(y_\Gamma-y_0)\sin\theta_i,\qquad 0\leqslant\theta_i<\pi,\qquad b=0,1,2,\qquad i=1,\ldots,n.\end{gathered} \tag{21}$$

Thus, we arrive at the following eigenfunctions and eigenvalues problem: given three independent homogeneous integral equations (21), it is required to find three spectra of eigenvalues $\{\nu_{bi}\}$ such that nontrivial solutions U_{bi} exist. To solve this problem, we partition the interval $[0,\pi)$ into small sectors $\Delta\theta_j$ $(j=1,\ldots,m)$ and write the integrals in (20) and (21) as finite sums. This approach allows one to find the average values of the functions $K_b(\theta_k^*)$ and $L_b(\theta_k^*)$ [14], which is sufficient for this method. Here θ_k^* are certain average values of the angles θ in the intervals

$\Delta\theta_k$. We assume that the partitioning is so fine ($m \gg n$) that each sector $\Delta\theta_k$ contains only one angle θ_i from the finite sums in (20). In this case, it is not known in advance which angles θ_i from the finite sums fall in the sectors $\Delta\theta_j$. To eliminate this uncertainty, we assume that the angles θ_i fall in each small sector $\Delta\theta_j$. If a certain sector does not contain angles θ_i from the finite sums, the corresponding quantities K^*_{bi} and L^*_{bi} vanish. Thus, we represent the functions U_b from (20) approximately by the sums

$$U_b = \sum_{j=1}^{m}[K_{bj}\cos(\nu_b\xi_j^{(b)}) + L_{bj}\sin(\nu_b\xi_j^{(b)})], \qquad 0 \leqslant \theta_j < \pi,$$

$$\begin{gathered} K_{bj}\cos\nu_b\xi_j^{(b)} = K_b(\theta_j^*)\cos\nu_b\xi_j^{(b)}(\theta_j^*)\Delta\theta_j + K^*_{bj}\cos\nu_b\xi_j^{(b)}, \\ L_{bj}\sin\nu_b\xi_j^{(b)} = L_b(\theta_j^*)\sin\nu_b\xi_j^{(b)}(\theta_j^*)\Delta\theta_j + L^*_{bj}\sin\nu_b\xi_j^{(b)}, \\ \xi_j^{(b)} = a_b(x-x_0)\cos\theta_j + c_b(y-y_0)\sin\theta_j, \end{gathered} \tag{22}$$

where θ_j^* are certain average values of the angles θ in the sectors $\Delta\theta_j$. In (22), the integral sums and finite sums with n terms are combined. Hence, the expressions for $K_{bj}\cos\nu_b\xi_j^{(b)}$ and $L_{bj}\sin\nu_b\xi_j^{(b)}$ in (22) consist of two parts. The first parts of the type $K_b(\theta_j^*)\cos\nu_b\xi_j^{(b)}(\theta_j^*)\Delta\theta_j$ (from the integral sums) depend on the partitioning method, whereas the second parts of the type $K^*_{bj}\cos\nu_b\xi_j^{(b)}$ (from the finite sums) are independent of this method. We use this property to find the quantities K^*_{bj} and L^*_{bj} and their number n in the finite sums in (20) and (21). If the order of certain K_{bj} and L_{bj} remains unchanged as $\Delta\theta_j$ decreases, the corresponding K^*_{bj} and L^*_{bj} exist and their number is equal to the desired n. In constructing the solution, it is not necessary to find the points D^+ and D^- at which the straight lines E intersect the boundary. A simpler method is to partition Γ into small intervals regardless of the angles θ_j and straight lines E, so that the number of the partition points on Γ is equal to $2m$, i.e., twice the number of angles θ_j, and satisfy the boundary conditions at these points. Setting $\boldsymbol{r}_\Gamma = \boldsymbol{r}_k$ ($k = 1, \ldots, 2m$) in (21), we write these boundary conditions as

$$\begin{gathered} U_b\Big|_\Gamma = \sum_{j=1}^{m}[K_{bj}\cos(\nu_b\xi_{kj}^{(b)}) + L_{bj}\sin(\nu_b\xi_{kj}^{(b)})] = 0, \qquad b = 0,1,2, \\ \xi_{kj}^{(b)} = a_b(x_k-x_0)\cos\theta_j + c_b(y_k-y_0)\sin\theta_j, \qquad 0 \leqslant \theta_j < \pi, \qquad k = 1,\ldots,2m. \end{gathered} \tag{23}$$

In (23), we have three independent linear algebraic homogeneous systems, each containing $2m$ equations for $2m$ unknowns K_{bj} and L_{bj}. These systems have nontrivial solutions if their determinants vanish:

$$\Delta_{2m}^{(b)} = |\cos(\nu_b\xi_{kj}^{(b)}), \sin(\nu_b\xi_{kj}^{(b)})| = 0, \qquad b = 0,1,2. \tag{24}$$

This is the desired characteristic equation for finding the independent spectra $\{\nu_{bi}, U_{bi}\}$. It was shown [15] that all roots of similar spectra ν_{bi} are real and distinct. Setting $\nu_b = \nu_{bi}$ in system (23), we infer that the determinants $\Delta_{2m}^{(b)}$ vanish. This implies that in each of the three systems (23), one equation becomes dependent on all remaining equations of the system and, hence, it can be omitted. All equations of the systems considered are equivalent; therefore, we drop, for example, the last equations and obtain three truncated linear systems with determinants $\Delta_{2m-1}^{(b)}$. For $\nu_b = \nu_{bi}$, the subscript i should be added to the coefficients K_{bj} and L_{bj} in systems (23), i.e., these coefficients are denoted as K_{bij} and L_{bij}. These coefficients from the truncated systems (23) are found with accuracy up to an arbitrary factor, which is determined below from the initial conditions. Therefore, one of these coefficients can be set equal to an unity, for example, $L_{bim} = 1$. Dropping one equation in each of the three systems (23), we obtain truncated systems each of which contains $(2m-1)$ equations for the same number of unknowns K_{bij} and L_{bij}:

$$\begin{gathered} \sum_{j=1}^{m} K_{bij}\cos[\nu^*_{bi}(\theta_j)\xi_{kj}] + \sum_{j=1}^{m-1} L_{bij}\sin[\nu^*_{bi}(\theta_j)\xi_{kj}] = -\sin[\nu^*_{bi}(\theta_m)\xi_{km}], \\ b = 0,1,2, \qquad i = 1,\ldots,\infty, \qquad k = 1,\ldots,(2m-1). \end{gathered} \tag{25}$$

From three systems (25), we find the coefficients K_{bij} and L_{bij}, substitute them into expressions (22), and obtain the desired eigenfunctions and eigenvalues $\{U_{bi}\}$:

$$U_{bi} = \sum_{j=1}^{m} K_{bij} \cos(\nu_{bi}\xi_j) + \sum_{j=1}^{m} L_{bij} \sin(\nu_{bi}\xi_j) \quad (b = 0, 1, 2, \quad i = 1, \ldots, \infty, \quad j = 1, \ldots, m). \tag{26}$$

Finding the spectra $\{\nu_{bi}, U_{bi}\}$ in explicit form (26) is an important point in constructing the solution of problem (2)–(5) in general form. For this, using (15) we write the solution of problem (8)–(10) as the sum

$$\begin{gathered} u^{(c)} = \sum_{i=1}^{\infty} (A_{ci} \cos(\nu_{ci}t) + B_{ci} \sin(\nu_{ci}t)) U_{ci}, \qquad c = 1, 2, \\ u^{(0)} = \sum_{i=1}^{\infty} A_{0i} U_{0i} \, \mathrm{e}^{-a\nu_{0i}^2 t}. \end{gathered} \tag{27}$$

By construction, the functions $u^{(c)}$ and $u^{(0)}$ from (27) satisfy system (8) and boundary conditions (9) at the points $\boldsymbol{r}_k$ for any A_{bi} and B_{ci}. We find these coefficients from the initial conditions (10), which become

$$\sum_{i=1}^{\infty} A_{bi} U_{bi} = \varphi_b, \qquad \sum_{i=1}^{\infty} \nu_{ci} B_{ci} U_{ci} = \psi_c, \qquad b = 0, 1, 2, \quad c = 1, 2. \tag{28}$$

Equalities (28) should be treated as Fourier expansions of the functions φ_b and ψ_c in series in the functional bases $\{U_{bi}\}$, and the quantities A_{bi} and B_{ci} as expansion coefficients. If $\varphi_b \in L_p^\alpha$ and $\psi_c \in L_p^\alpha$ (L_p^α are the classes of Sobolev–Liouville functions [15]), series (28) converge uniformly in Ω [15] and their expansion coefficients are given by

$$A_{bi} = \frac{1}{N_{bi}} \iint\limits_{\Omega} \varphi_b U_{bi} \, ds, \quad B_{ci} = \frac{1}{\nu_{ci} N_{ci}} \iint\limits_{\Omega} \psi_c U_{ci} \, ds, \quad N_{bi} = \iint\limits_{\Omega} U_{bi}^2 \, ds \quad (b = 0, 1, 2, \quad c = 1, 2). \tag{29}$$

Thus, the solution of problem (8)–(10) is constructed as follows: the functions $u^{(b)}$ ($b = 0, 1, 2$) should be taken from (27), the coefficients A_{bi} and B_{ci} from (29), the eigenfunctions U_{bi} from (26), and the eigenvalues ν_{bi} from the solution of the characteristic equation (24). We now proceed to the solution of problem (11)–(13).

4. Constructing the Solution for the Displacements of the Body. Since the three bases $\{U_{bi}\}$ ($b = 0, 1, 2$) belong to the class $C^\infty(\Omega)$ and are independent of one another, each eigenfunction (or its partial derivatives with respect to x and y) in one basis can be expanded in terms of the eigenfunction of another basis. Below, we use the spectral decompositions

$$U_{2ixy} = \sum_{j=1}^{p} \alpha_{ij}^{(1)} U_{1i}, \quad U_{1ixy} = \sum_{j=1}^{p} \alpha_{ij}^{(2)} U_{2j}, \quad \alpha_{ij}^{(1)} = \frac{1}{N_{1j}} \iint\limits_{\Omega} U_{2ixy} U_{1j} \, ds, \quad \alpha_{ij}^{(2)} = \frac{1}{N_{2j}} \iint\limits_{\Omega} U_{1ixy} U_{2j} \, ds; \tag{30}$$

$$U_{1ix} = \sum_{j=1}^{p} \gamma_{ij}^{(1)} U_{0j}, \quad U_{2iy} = \sum_{j=1}^{p} \gamma_{ij}^{(2)} U_{0j}, \quad \gamma_{ij}^{(1)} = \frac{1}{N_{0j}} \iint\limits_{\Omega} U_{1ix} U_{0j} \, ds, \quad \gamma_{ij}^{(2)} = \frac{1}{N_{0j}} \iint\limits_{\Omega} U_{2iy} U_{0j} \, ds; \tag{31}$$

$$G_c = \sum_{j=1}^{p} g_{cj} U_{cj}, \qquad g_{cj} = \frac{1}{N_{cj}} \iint\limits_{\Omega} G_c U_{cj} \, ds, \qquad c = 1, 2; \tag{32}$$

$$\begin{gathered} U_{0ix} = \sum_{j=1}^{p} \alpha_{ij}^{(0)} U_{1i}, \qquad U_{0iy} = \sum_{j=1}^{p} \beta_{ij}^{(0)} U_{2i}, \qquad q = \sum_{i=1}^{p} q_i U_{0i}, \\ \alpha_{ij}^{(0)} = \frac{1}{N_{1j}} \iint\limits_{\Omega} U_{0ix} U_{1j} \, ds, \qquad \beta_{ij}^{(0)} = \frac{1}{N_{2j}} \iint\limits_{\Omega} U_{0iy} U_{2j} \, ds, \qquad q_i = \frac{1}{N_{0i}} \iint\limits_{\Omega} q U_{0i} \, ds; \end{gathered} \tag{33}$$

$$U_{1ixx} = \sum_{j=1}^{p} \beta_{ij}^{(1)} U_{1j}, \quad U_{2iyy} = \sum_{j=1}^{p} \beta_{ij}^{(2)} U_{2j}, \quad \beta_{ij}^{(1)} = \frac{1}{N_{1j}} \iint\limits_{\Omega} U_{1ixy} U_{1j} \, ds, \quad \beta_{ij}^{(2)} = \frac{1}{N_{2j}} \iint\limits_{\Omega} U_{2iyy} U_{2j} \, ds. \tag{34}$$

The spectral decompositions of $U_{1iyy}$and U_{2ixx} can be obtained using the equations from system (15) and the expansions for U_{1ixx} and U_{2iyy} from (34):

$$U_{1iyy} = -\frac{\rho}{\mu}\,\nu_{1i}^2 U_{1i} - \Big(\frac{\lambda}{\mu} + 2\Big)\sum_{j=1}^{p} \beta_{ij}^{(1)} U_{1j}, \quad U_{2ixx} = -\frac{\rho}{\mu}\,\nu_{2i}^2 U_{2i} - \Big(\frac{\lambda}{\mu} + 2\Big)\sum_{j=1}^{p} \beta_{ij}^{(2)} U_{2j}. \tag{35}$$

The number of terms p in (30)–(35) is determined by the calculation accuracy. The need for these formulas is due to the fact that the equations of system (11) are coupled. We seek a solution of problem (11)–(13) in the form

$$v^{(b)} = \sum_{i=1}^{p} T_{bi}(t) U_{bi}, \qquad b = 0, 1, 2, \tag{36}$$

where $T_{bi}(t)$ are unknown functions of the single variable t. To find them, we substitute the expressions for $v^{(b)}$ from (36) into system (11):

$$\begin{gathered}
a\sum_{i=1}^{p} T_{0i}\Delta U_{0i} - \eta\sum_{i=1}^{p}[T_{1i}' U_{1ix} + T_{2i}' U_{2ix}] + q = \sum_{i=1}^{p} T_{0i}' U_{0i},\\
(\lambda+2\mu)\sum_{i=1}^{p} T_{1i} U_{1ixx} + (\lambda+\mu)\sum_{i=1}^{p} T_{2i} U_{2ixy} + \mu\sum_{i=1}^{p} T_{1i} U_{1iyy} + (\zeta+2\chi)\sum_{i=1}^{p} T_{1i}' U_{1ixx}\\
+ (\zeta+\chi)\sum_{i=1}^{p} T_{2i}' U_{2ixy} + \chi\sum_{i=1}^{p} T_{1i}' U_{1ixx} - \gamma\sum_{i=1}^{p} T_{0i} U_{0ix} + G_1 = \rho\sum_{i=1}^{p} T_{1i}'' U_{1i},\\
(\lambda+2\mu)\sum_{i=1}^{p} T_{2i} U_{2iyy} + (\lambda+\mu)\sum_{i=1}^{p} T_{1i} U_{1ixy} + \mu\sum_{i=1}^{p} T_{2i} U_{2ixx} + (\zeta+2\chi)\sum_{i=1}^{p} T_{2i}' U_{2iyy}\\
+ (\zeta+\chi)\sum_{i=1}^{p} T_{1i}' U_{1ixy} + \chi\sum_{i=1}^{p} T_{2i}' U_{2ixx} - \gamma\sum_{i=1}^{p} T_{0i} U_{0iy} + G_2 = \rho\sum_{i=1}^{p} T_{2i}'' U_{2i}.
\end{gathered} \tag{37}$$

Each term in the first, second, and third equations of system (37) should be expanded in terms of the eigenfunctions $U_0(\nu_{0i})$, $U_1(\nu_{1i})$, and $U_2(\nu_{2i})$, respectively. To this end, we use Eqs. (16) and the spectral decompositions (30)–(35). Then, equating the coefficients of the corresponding eigenfunctions, we obtain the following system of ordinary differential equations for $T_{bi}(t)$ $(b = 0, 1, 2)$:

$$\begin{gathered}
T_{0i}' + \eta\sum_{j=1}^{p}(\gamma_{ji}^{(1)} T_{1j}' + \gamma_{ji}^{(2)} T_{2j}') + a\nu_{0i}^2 T_{0i} = q_i, \qquad i = 1, \ldots, p,\\
\rho T_{1i}'' + \rho\nu_{1i}^2 T_{1i} - (\lambda+\mu)\sum_{j=1}^{p}\alpha_{ji}^{(1)} T_{2j} - (\zeta+2\chi)\sum_{j=1}^{p}\beta_{ji}^{(1)} T_{1j}' - (\zeta+\chi)\sum_{j=1}^{p}\alpha_{ji}^{(1)} T_{2j}'\\
+ \chi\,\frac{\rho}{\mu}\,\nu_{1i}^2 T_{1i}' + \chi\Big(\frac{\lambda}{\mu}+2\Big)\sum_{j=1}^{p}\beta_{ji}^{(1)} T_{1j}' + \gamma\sum_{j=1}^{p}\alpha_{ji}^{(0)} T_{0j} = g_{1i},\\
\rho T_{2i}'' + \rho\nu_{2i}^2 T_{2i} - (\lambda+\mu)\sum_{j=1}^{p}\alpha_{ji}^{(2)} T_{1j} - (\zeta+2\chi)\sum_{j=1}^{p}\beta_{ji}^{(2)} T_{2j}' - (\zeta+\chi)\sum_{j=1}^{p}\alpha_{ji}^{(2)} T_{1j}'\\
+ \chi\,\frac{\rho}{\mu}\,\nu_{2i}^2 T_{2i}' + \chi\Big(\frac{\lambda}{\mu}+2\Big)\sum_{j=1}^{p}\beta_{ji}^{(2)} T_{2j}' + \gamma\sum_{j=1}^{p}\beta_{ji}^{(0)} T_{0j} = g_{2i}.
\end{gathered} \tag{38}$$

Using (13), we obtain the following initial conditions for the inhomogeneous system (38):

$$T_{bi}\Big|_{t=0} = 0, \quad b = 0, 1, 2; \qquad T_{ci}'\Big|_{t=0} = 0, \quad c = 1, 2. \tag{39}$$

The linear system of ordinary differential equations (39) consists of $3p$ equations for $3p$ unknowns T_{0i}, T_{1i}, and T_{2i}. Its solution subject to the initial conditions (39) can be written in terms of quadratures by using, for example, the method of variation of constants [16].

We assume that system (38) is solved subject to the initial conditions (39) and the functions T_{bi} ($b = 0, 1, 2$, $i = 1, \ldots, p$) are found. Using $T_{bi}(t)$ and U_{bi}, we determine $v^{(b)}$ from formulas (36). Substitution of $u^{(b)}$ and $v^{(b)}$ into (6) yields the desired solution of the problem (2)–(5).

5. Error Estimate. The error of the method is due to the replacement of integrals in (20) and (21) by the integral sums (22) and the satisfaction of the boundary conditions only at the calculation points rather than everywhere on Γ. The integrals are replaced by integral sums using the trapezoidal formula, Simpson's rule, etc. [17]. Each of these formulas can be written as a linear relation between the values of the integrand function at the calculation points of the variable θ. This relation can be reduced to the form (22) with accuracy up to notation. Denoting the difference between the exact solution U and the approximate solution $\tilde{U}$ by $\delta U = U - \tilde{U}$ and using Simpson's rule, we obtain

$$\delta U = \Delta\theta^5 W(x, y), \qquad (x, y) \in \Omega, \qquad \Delta\theta = \max \Delta\theta_k \tag{40}$$

[$W(x, y)$ is a limited function in Ω]. Since $U \in C^\infty(\Omega)$ and $\tilde{U} \in C^\infty(\Omega)$, then $W \in C^\infty(\Omega)$. To estimate the function W, we use a pattern of two points D_i and D_{i+1} on Γ. We introduce a local coordinate system in such a manner that the x axis passes through these points and the y axis passes through the midpoint of the segment D_iD_{i+1}. The equation of the part of the boundary D_iD_{i+1} is written as

$$y = y(0) + y'(0)x + y''(0)x^2/2 + O(h^3) \tag{41}$$

($2h$ is the distance between the neighboring partition points on Γ). Using the condition $y(\pm h) = 0$, we simplify Eq. (41):

$$y = æ(x^2 - h^2)/2 + O(h^3), \qquad æ = y''(0). \tag{42}$$

We expand W in a Taylor series in the neighborhood of the coordinate origin, restrict ourselves to terms linear in y and quadratic in x:

$$W = W(0, 0) + W_x x + W_{xx}x^2/2 + W_{xy}xy + W_y y + \ldots \ . \tag{43}$$

In (42), the expansion for y contains terms up to x^2; therefore, in (43) terms linear in y and quadratic in x should be retained as terms of the same order of smallness. Using the relation $W(D_i) = W(D_{i+1}) = 0$, we simplify expansion (43):

$$W = W_{xx}(0, 0)(x^2 - h^2)/2 + W_y(0, 0)y + \ldots \ . \tag{44}$$

In (44), the parabolic dependence of W on x implies that $\max |W|$ is reached on Γ at the midpoint between the points D_i and D_{i+1}, i.e., at the point $Q(0, -æh^2/2)$:

$$\max |W| = (1/2)h^2 \Big| W_{xx} + æW_y \Big|_\Gamma + \ldots \ .$$

From this, taking into account (40), we obtain

$$|\delta U| = (1/2)h^2\Delta\theta^5 \Big| W_{xx} + æW_y \Big|_\Gamma + \ldots \ .$$

Denoting the maximum diameter of the domain Ω by D_0, we have $D_0\Delta\theta > 2h$ and finally obtain the error estimate:

$$|\delta U| < (1/8)D_0^2\Delta\theta^7 \Big| W_{xx} + æW_y \Big|_\Gamma + \ldots \ . \tag{45}$$

In particular, estimate (45) shows that the size h should be diminished as the curvature $æ$ of the boundary Γ increases.

6. Numerical Experiment. Numerical experiments were performed to compare the approximate solution and the exact solution for a unit circle. Using a linear transformation, the differential operator on the left side of (16) can be reduced to the Laplace operator, for which a numerical experiment was performed. The error of the method arises first in determining the eigenfunctions $U_{bi} = U_i$ from (22) for $\nu_b = \nu_{bi} = \nu_i$ and then in the subsequent spectral decompositions. Therefore, we can confine our considerations to the errors in calculating U_i

TABLE 1

m	$\delta\nu_1$	$\delta\nu_2$	$\delta\nu_i$
20	10^{-45}	10^{-30}	$\max i = 8$; $\delta\nu_8 = 10^{-6}$
30	10^{-77}	10^{-55}	$\max i = 13$; $\delta\nu_{13} = 10^{-6}$
50	10^{-150}	10^{-114}	$\max i = 24$; $\delta\nu_{24} = 10^{-6}$

TABLE 2

m	δU_1	δU_2	δU_{1x}	δU_{2x}	δU_{1xx}	δU_{2xx}	δU_{1xxx}	δU_{2xxx}
20	10^{-45}	10^{-30}	10^{-43}	10^{-29}	10^{-42}	10^{-27}	10^{-39}	10^{-24}
50	10^{-150}	10^{-114}	10^{-148}	10^{-112}	10^{-146}	10^{-110}	10^{-142}	10^{-106}

and ν_i. The errors in the eigenfunction δU_i and their partial derivatives were determined at the midpoints between the partition points on Γ. At approximately these points, the error δU_i reaches a maximum value.

The errors $\delta\nu_i$ for various m (m is the number of straight lines E) are listed in Table 1. One can see that 24 eigenfunctions and eigenvalues U_i and ν_i are calculated with high accuracy for 50 radials ($m = 50$) and 100 points on Γ. The errors in determining U_i and ν_i are very small, whereas the errors δU_i and $\delta\nu_i$ increase monotonically with the number i. This drawback is partly eliminated since, by virtue of convergence of the spectral decompositions, the effect of the eigenfunction U_i decreases as i increases. It can be concluded that for $2m$ ($m > 30$) partition points on Γ, the method allows one to calculate approximately $m/2$ eigenvalues and eigenfunctions with an error not worse than 10^{-6}. As m increases, all errors decrease rapidly. The error for $\delta\nu_1 \approx 10^{-3m}$. The choice of the value of m depends not only on the errors δU_i but also on the number of terms in the spectral decompositions required to provide the specified accuracy. In this method, partial derivatives can be calculated analytically. The errors for the partial derivatives at the boundary points on the x axis are given in Table 2. It follows that each two derivatives increase the error by approximately three orders of magnitude. Nevertheless, the errors even for the fourth partial derivative are very small. The location of the pole $\boldsymbol{r}_0$ has little effect on the calculation error.

In finite-difference methods for a unit circle using a 100-node mesh, the error is much higher and the calculations should be repeated many times in each small time layer, which involves sub-boundary difficulties and increases the error.

We note that for the approximate solution of the initial boundary-value problem, the boundary conditions are satisfied approximately between the calculation points on Γ but these conditions are satisfied exactly at the calculation points. The approximate solution has analytic form, which allows one to perform various model experiments and studies. Moreover, it satisfies the initial system of differential equations and, hence, adequately describes the physical properties used in the mathematical model. The initial conditions (5) can be satisfied with any specified accuracy by retaining an appropriate number of terms p in decompositions (30)–(35). For 100 partition point on the boundary, it suffices to solve a linear system of 100 equation once. As a result, one obtains a solution which can be used for any finite interval of time. Particular solutions obtained by this method for simpler models can be found, for example, in [10, 18].

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